The slope of this line at $\Omega = 90$ deg is proportional to $Mc/EA_g$, i.e., larger $M$ or smaller $A_e$ makes for a more rapid transition of the top boundary condition from large to small values of $\Omega$ for constant values of $M/k$ and vice versa. Two curves for the top boundary condition are shown in Fig. 6 corresponding to two values of the effective spring constant $k$, differing by a factor of 2.43 (the values are the experimental and estimated values mentioned previously). It will be seen that the change in $k$ has little effect on the natural frequencies except near the natural frequency of the system supporting the drill pipe (derrick, drilling lines, traveling block, and so forth). This is a convenient result for in practice the value of $k$ will vary appreciably (by a factor of 1.5, say) as the traveling block moves between its extreme positions.

**Discussion**

J. E. Goldberg and J. L. Bogdanoff

It is common knowledge that the dynamical behavior of a drill pipe in rotary drilling is extremely complicated, being determined by the method of drive, the forces acting on the bit, and by the forces arising from contact of the pipe with the sides of the hole or with pieces of rock between the pipe and the sides of the hole. In particular, the time behavior of the forces at the bit and on the side of the pipe cannot be described in any deterministic sense because of their random or chance nature. These points suggest that a statistical description of drill pipe dynamics might be fruitful.

We presented a paper entitled “A New Analytical Approach to Drill Pipe Bending” at the September, 1958, Denver meeting of the ASME Petroleum Mechanical Engineering Conference. The central purpose of the paper was to exhibit the advantages of a statistical approach to the very complex problem of drill pipe dynamics and to recommend an experimental program to secure necessary statistical information or input data with which this new approach could be put on a quantitative basis. A certain aspect of drill pipe dynamics is discussed by the present authors from a deterministic point of view. It is therefore of considerable interest to contrast the two methods of approach. It is necessary, however, to first describe the system considered by us inasmuch as our paper is not readily available.

The model considered in our paper is shown in Fig. 7. For purposes of preliminary discussion, only longitudinal displacement $u(x, t)$ and torsional displacement $\Theta(x, t)$ were allowed. Gravity was neglected, viscous damping was assumed, and the pipe was taken to be uniform in cross-section; this leads to the equations of motion:

$$
\frac{\partial^2 u}{\partial t^2} + 2b_\alpha \frac{\partial u}{\partial t} = a_\alpha \frac{\partial^2 u}{\partial x^2}
$$

$$
\frac{\partial^2 \Theta}{\partial t^2} + 2b_r \frac{\partial \Theta}{\partial t} = a_r \frac{\partial^2 \Theta}{\partial x^2}
$$

The boundary conditions at the left end, i.e., at the top of the hole, depend upon the method of drive, method of support, etc., and in each problem may be estimated fairly easily. We took for purposes of illustration only

$$
AE \frac{\partial u}{\partial x} \bigg|_{x=0} = k_u \Theta(0, t)
$$

$$
C \frac{\partial \Theta}{\partial x} \bigg|_{x=0} = k_\Theta \Theta(0, t)
$$

which represent the conditions for an elastic connection.

We assumed that a random axial force $F(l, t)$ and a random couple $T(l, t)$ to be second-order mean square continuous, weakly stationary random functions with zero means

$$
E\{F(l, t)\} = 0, \quad E\{T(l, t)\} = 0
$$

and covariance

$$
\Gamma_{TF}(t_1 - t_2) = E\{T(l, t) F(l, t_2)\}
$$

The boundary conditions at the lower end were therefore

$$
AE \frac{\partial u}{\partial x} \bigg|_{x=l} = F(l, t)
$$

$$
C \frac{\partial \Theta}{\partial x} \bigg|_{x=l} = T(l, t)
$$

It might seem at first sight that $u(x, t)$ and $\Theta(x, t)$ do not depend upon one another. This would appear to follow from the fact that the equations of motion and boundary conditions for the two displacements are separate. However, $F(l, t)$ and $T(l, t)$ are related statistically through (12). Hence $u(x, t)$ and $\Theta(x, t)$ are not independent of one another in our analysis. Our views on the boundary conditions at the top of the hole coincide with the authors'; i.e., they may be estimated with fair accuracy for each type of drive. While these conditions may be complex in certain circumstances, they can be handled without undue difficulty.

We have assumed an axial force and torque (random, however) acting at the bottom of the pipe. This assumption leads to natural frequencies, when damping is neglected, which correspond to a free end. The authors, on the other hand, assume that the string is fixed longitudinally and free torsionally at the bottom. The assumption that longitudinal displacement is zero implies the presence of a rigid constraint and neglects elasticity of the rock, of the bit, and of the possibility of the bit bouncing up from the rock which is supposed to be cutting. If one really wants an accurate estimate of natural frequencies, such factors should not be ignored.

The natural frequencies in undamped linear systems provide only partial information about the dynamical behavior of such systems. If, for example, there are simple harmonic components in the exciting force system which coincide with one or more of the natural frequencies and do not act at nodes of the corresponding modes, the phenomenon of resonance occurs. Amplitudes may not be estimated, however, unless damping is assumed. Moreover, if the actual damping present is large, undamped natural frequencies are at best only rough guides to frequencies of oscillation. As is well known, a complete description of dynamical behavior of a damped linear system is contained in the frequency response (responses if there is more than one point of excitation).
The frequency response in longitudinal motion which we obtained is

\[ G_A(x, i\omega) = \frac{1}{AE} \frac{\gamma \cosh \frac{\gamma x}{l} + \sinh \frac{\gamma x}{l}}{\frac{\gamma^2}{\alpha} \sinh \gamma + \gamma \cosh \gamma} \]

where

\[ \alpha = \frac{k_A}{AE} \]

\[ \gamma = \frac{it}{a_A} \sqrt{\omega^2 - 2ib_0\omega} \]

\[ |G_A(x, i\omega)| \] becomes large when \( \omega \) is such that

\[ \frac{\gamma}{\alpha} \sinh \gamma + \cosh \gamma \]

is small compared to the absolute value of the numerator. For \( b_A = 0 \), i.e., for the undamped case, this occurs when \( \omega \) satisfies the equation

\[ \tan \frac{\omega l}{a_A} = \frac{k_A}{AE} \frac{1}{\omega l} \]

and these \( \omega \) are the undamped longitudinal natural frequencies of our model; they do not correspond to those obtained by the authors. As \( b_A \) increases from zero, the values of \( \omega \) satisfying (17) depart further and further from the values of \( \omega \) which make \( |G_A(x, i\omega)| \) large. Because of the possibility of rubbing between the pipe and wall, the presence of internal damping, the presence of slipping in tool joints, etc., it is reasonable to expect substantial damping in drill strings. In addition, the length of a string changes as cutting progresses. Hence undamped natural frequencies begin to assume a somewhat academic aspect even if a deterministic point of view is adopted in studying drill pipe dynamics.

These remarks also imply that a visual search for a set of natural frequencies based on the assumption of no damping and definite boundary conditions in records such as the authors present, which appear to be random, is hardly adequate. What is needed is a power spectral density analysis of these records. Sharp peaks in these densities would indicate either dominant exciting frequencies or natural frequencies, if any of the latter are present.

Many of the quantities of interest in our paper are of the form

\[ \int_0^\infty |H(x, i\omega)|^2 P(l, \omega) d\omega \]

where, for example,

\[ H_A(x, i\omega) = AE \frac{\partial G_A}{\partial x} \]

\[ P_A(l, \omega) = \text{power spectral density of axial bit force} \]

The peaks of \( |H_A(x, i\omega)|^2 \) as a function of \( \omega \) determine the regions of resonance. The frequencies at which these peaks occur are determined by the boundary conditions, physical constants, and damping. Their magnitudes, if large, are primarily controlled by damping. If \( P_A(l, \omega) \) is a slowly varying function of \( \omega \), the exact positions of these peaks will not appreciably influence the value of (18), provided the magnitudes of the peaks are about the same. If the peaks are not large, the same remark is valid even if \( P_A(l, \omega) \) varies somewhat more rapidly as a function of \( \omega \). Hence (18) is not unduly sensitive by changes in boundary conditions and damping, if the latter is reasonably large. Thus our analysis is much less sensitive on the whole to doubtful or uncertain boundary conditions than a deterministic analysis.

Since the very essence of a statistical analysis is the accommodation of uncertainty (in loads in our analysis, but uncertainty in boundary conditions and damping could also be included), its range of application in highly complex problems involving random loads, etc., is substantial and the advantages it offers in such problems is worthy of consideration.

**Authors’ Closure**

The authors wish to thank Professors Goldberg and Bogdanoff for their description of their paper and their stimulating discussion. Statistical approaches have been used with success to predict results based on past measurement in a number of technological areas. In particular, the theory of random vibrations has received considerable attention in aircraft and missile applications, where vibration caused by excitation due, for example, to jet and rocket engine noise, is logically treated as a statistical problem.

Clearly the objectives of our paper are quite different from those of the discussors, for the present paper concentrates on the determination of the lower natural frequencies of an idealized model of a drill string and the manner in which certain changes in the system alter the natural frequencies.

We do not wish to underestimate the practical difficulties of our assumptions, indeed we have already pointed many out. However, the degree to which the assumptions are reasonable will vary greatly from hole to hole, depending on such factors as the hole depth, the crookedness of the hole, the type of drilling fluid used, the nature of the buckling in the collars, and so forth. All these factors will also complicate a statistical analysis.

The discussers point out that their equation (17) does not agree with our own results. The reason, of course, is that they have assumed different boundary conditions from those that we used. In particular, they have assumed the bottom end of the string effectively free to longitudinal vibrations. Our best present analyses do not support such a result; nor yet do we feel it likely that the bit, carrying mean loads approaching 100,000 lb, should, in any ordinary circumstances, continually bounce clear of the rock.

In summary, then, we agree with Professors Goldberg and Bogdanoff that a treatment of the drill-string vibration problem by the techniques of random vibration theory might well prove interesting. We must also agree with them that (as they pointed out in their original paper) significant calculations by these techniques are out of the question at the moment, as the basic experimental information necessary is not available.